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Dynamical potential algebras for Gendenshtein and Morse potentials

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Abstract. A differential realization of so(2, 1) is shown to be the potential algebra for the one-dimensional systems with the Morse or Gendenshtein potentials. This shows that two classes of Gendenshtein potentials will support the same eigenvalues as the Morse potential, and that the three sets of eigenfunctions may be derived in a common formalism. The potential algebra is then extended to a dynamical potential algebra with operators connecting states both in different potentials and with different energies, giving new dynamical algebras for the Gendenshtein problem. The matrix elements of certain corresponding operators in the three types of system may then be given by a single formula involving so(2, 1) Wigner coefficients. We also give ladder operators connecting the Gendenshtein potential eigenstates.

1. Introduction

The Morse potential is one of the one-dimensional quantum mechanical systems for which the eigenvalues can be determined by an algebraic method, and the eigenfunctions (bound states) can be generated from the ground state using operators that realise a Lie algebra (Cordero and Hojman 1970, Berrondo and Palma 1980, Alhassid *et al* 1983). When the operators actually produce eigenfunctions belonging to the same eigenvalue of systems with different potential parameters, the algebra is a potential algebra, and the algebraic method gives results for a family of potentials (Alhassid *et al* 1986). This is essentially a group-theoretical interpretation of the use of supersymmetry, or the Darboux transformation. The potentials of the related systems are called shape-invariant (Gendenshtein 1983) when they have the same functional form but different values of some numerical parameters. Gendenshtein's search for such potentials revealed new soluble problems with the Gendenshtein potentials, given here in (6.4) and (6.6). Wavefunctions have been obtained (Dabrowska *et al* 1988) using the potential algebra operators, and also (Pertsch 1990) by the usual method of using special functions to solve the Schrödinger equation. This may well be the only example where the algebraic solution has preceded analytic methods.

Algebraic methods were first developed by taking some particular system and looking for suitable associated operators. Gendenshtein took the different approach of requiring shape invariance, and then determining potentials that satisfied the

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necessary conditions. A similar procedure (Cordero *et al* 1971, Wu and Alhassid 1990) determines some operators which form a realization of an algebra or group, in particular so(2, 1), and then obtains the potentials that appear in associated Schrödinger equations for which the realization is the potential algebra. This procedure produced the Morse potential and two special cases of Pöschl-Teller potentials, for which associated algebras were already known. We have found that Wu and Alhassid did not use the most general realization of so(2, 1) allowed by their assumptions and that, when this generalization is made, the associated potentials include two classes of Gendenshtein potentials. These potentials, together with the Morse potential, may be treated in a common formalism, explaining why they have the same eigenvalues. The common properties of these three classes extend to the calculation of eigenfunctions, and the evaluation of certain matrix elements.

As well as showing that these apparently different systems have the same potential algebra, we have also obtained for them dynamical algebras of operators connecting states of different energy. Previously so(4, 2) has been used as a dynamical potential algebra for the Morse potential (Barut *et al* 1987). We get the algebras so(3, 1), so(2, 2) and iso(2, 1) for the three classes, which may also be distinguished by whether the value of a certain Casimir operator is positive, negative or zero. Dynamical algebras for the Gendenshtein potentials have not been obtained before.

In the following section of this paper we give the realization of so(2, 1), and the Schrödinger potentials that appear when the realization is considered as a potential algebra, in terms of two functions that satisfy coupled ordinary differential equations. These equations are easily solved, but most of our results can be obtained leaving the explicit form of these solutions unspecified. The potential algebra provides expressions for the eigenfunctions, and relations between the constants required to normalize them. In section 3 the dynamical potential algebras are obtained, and their Casimir operators evaluated.

Next we get the representations of the algebras, including the matrix elements of operators forming a standard basis of the algebra. These matrix elements are physically significant when they are between eigenfunctions in the *same* potential. In section 5 these physical results are picked out. Our results could be obtained by manipulating commutators, but simple procedures of this type only show which matrix elements are zero.

At this stage all the work is in terms of the two unspecified solutions of differential equations. In section 6 the solutions are obtained, and classified to give the three types of potential. The asymptotic behaviour of the functions then shows whether the normalizable eigenfunctions and matrix elements actually exist.

It is also possible to obtain ladder operators that generate different energy eigenfunctions in the same potential, treating the three types of potential together. The details are given in section 7. Our results combine some unpublished results (Fellemans 1989) on the Gendenshtein potential with well known results in the Morse case (Nieto and Simmons 1979, Berrondo and Palma 1980).

The following summary provides a convenient reference to the main physical results of the paper. The Schrödinger equation (2.6) is considered with the potentials given in (6.4), (6.5) and (6.6), for which (6.8) gives the bound state eigenvalues. In conjunction with table 3, (5.2)-(5.4) and (6.1)-(6.3) give normalized eigenfunctions. Equations (5.12)-(5.14) give the matrix elements of the operators in table 4. This presentation of the results emphasizes the fact that from our algebraic viewpoint these problems are different realizations of a single underlying structure.

2. The potential algebra so(2, 1)

The operators

$$J_0=-\mathrm{i}\frac{\partial}{\partial\phi} \qquad J_\pm=\mathrm{e}^{\pm\mathrm{i}\phi}\left[\pm\frac{\partial}{\partial x}+f(x)\left(\mathrm{i}\frac{\partial}{\partial\phi}\mp\frac{1}{2}\right)+g(x)\right] \tag{2.1}$$

are a realization of the algebra so(2, 1) if the functions f and g satisfy the differential equations (Wu and Alhassid 1990)

$$f'=1-f^2 \qquad g'=-fg. \tag{2.2}$$

Then we have the commutation relations

$$[J_+,J_-]=-2J_0 \qquad [J_0,J_\pm]=\pm J_\pm. \tag{2.3}$$

With real f and g, $J_\pm$ are Hermitian conjugates relative to an inner product for which $(\partial/\partial x)^\dagger=-(\partial/\partial x)$ and $(\partial/\partial\phi)^\dagger=-(\partial/\partial\phi)$. The Casimir operator $J^2=(J_0^2\mp J_0-J_\pm J_\mp)$ has the form

$$J^2=\frac{\partial^2}{\partial x^2}-f'\left(\frac{\partial^2}{\partial\phi^2}+\tfrac{1}{4}\right)+2\mathrm{i}g'\frac{\partial}{\partial\phi}-g^2-\tfrac{1}{4}. \tag{2.4}$$

If $|k\,m\rangle=\psi_{km}(x)\,\mathrm{e}^{\mathrm{i}m\phi}$ are basis functions for the representation space of an so(2, 1) irreducible representation of type D_k^+, i.e.

$$J_0|k\,m\rangle=m|k\,m\rangle \qquad J^2|k\,m\rangle=k(k-1)|k\,m\rangle \qquad (m=k,k+1,k+2,\ldots) \tag{2.5}$$

then (Wu and Alhassid 1990) equations (2.5) require ψ_{km} to satisfy the Schrödinger equation

$$-\psi''+V_m\psi=-(k-\tfrac{1}{2})^2\psi \tag{2.6}$$

where the potential is

$$V_m=(\tfrac{1}{4}-m^2)f'+2mg'+g^2. \tag{2.7}$$

Thus, an irreducible representation of the potential algebra so(2, 1), associated with potentials of the form (2.7), has basis states which are eigenfunctions of different Hamiltonians but belong to the same energy eigenvalue. This eigenvalue is determined by the value of the Casimir operator (2.4): (2.6) is just the eigenvalue equation for $-J^2-\frac{1}{4}$ with $\partial/\partial\phi$ in (2.4) replaced by $\mathrm{i}m$.

Wu and Alhassid have considered Morse, Pöschl–Teller and Rosen–Morse potentials which are cases of (2.7) obtained using particular solutions of (2.2). One of our objectives is to extend this to the Gendenshtein potentials (Gendenshtein 1983) by using more general solutions of (2.2). However, we find that much of our work can be carried through using only (2.2) rather than the explicit forms of their solutions. We therefore do not need to specify these forms until a later section of this paper.

The functions ψ_{km} may be obtained by solving $J_-\psi_0(x)\,\mathrm{e}^{\mathrm{i}k\phi}=0$ for $\psi_0=\psi_{kk}$, and then calculating $\psi_n=\psi_{kk+n}$ by evaluating $J_+^n\psi_0(x)\,\mathrm{e}^{\mathrm{i}k\phi}$. The equation for ψ_0 is

$$\psi_0'=(\tfrac{1}{2}f-kf+g)\psi_0 \tag{2.8}$$

which, using $f=-g'/g$, integrates to

$$\psi_0=g^{k-1/2}h=g^{k-1/2}\exp\left(\int g(x)\,\mathrm{d}x\right). \tag{2.9}$$

Using equation (2.8),

$$e^{i(-k-1)\phi} J_+\psi_0 e^{ik\phi} = 2(g-kf)g^{k-1/2}h \tag{2.10}$$

and

$$e^{i(-k-2)\phi} J_+^2\psi_0 e^{ik\phi} = 2[2g^2-(4k+2)fg+(2k^2+2k)f^2-k]g^{k-1/2}h. \tag{2.11}$$

The physical application of these results lies in taking these functions as the first three eigenfunctions of the potential (2.7), and it is then desirable to include normalization constants. If (2.9) is normalized by multiplying by N_k, this constant cannot be calculated algebraically, but the standard relation

$$J_+|k\,m\rangle = [(m+k)(m-k+1)]^{1/2}|k\,m+1\rangle$$

between normalized states determines the factors to be included in the ψ_n for $n>0$. These are $(2k)^{-1/2}$ for (2.10) and $\frac{1}{2}(2k^2+k)^{-1/2}$ for (2.11). Note again that these wavefunctions are solutions of (2.6) with different potentials: (2.8), (2.9) and (2.10) will be solutions of (2.6) and (2.7) with $m=k$, $k+1$ and $k+2$ respectively.

A recurrence relation for the (ground state) normalization constants N_k will now be obtained. Suppose (α,∞) is the domain of the wavefunctions. (In section 6 we will take α to be either $-\infty$ or 0.) Then

$$N_k^{-2} = \int_\alpha^\infty g^{2k-1}h^2\,dx = \tfrac{1}{2}\int_\alpha^\infty g^{2k-2}\,d(h^2)$$

since $h'=gh$. After integrating by parts twice we reach

$$N_k^{-2} = \tfrac{1}{2}(k-1)\int_\alpha^\infty g^{2k-3}[(2k-2)f^2-1]h^2\,dx \tag{2.12}$$

provided

$$f^c g^{2k-2-c}h^2 \tag{2.13}$$

is zero at $x=\alpha$ and at $x=\infty$ for $c=0$ and for $c=1$. From (2.2) it is easy to verify that

$$\frac{d}{dx}\left(\frac{f^2-1}{g^2}\right)=0. \tag{2.14}$$

We may therefore put $f^2=1+\lambda g^2$ in (2.12), where λ is a constant, and thus obtain the recurrence relation

$$N_k^{-2}[1-\lambda(k-1)^2] = N_{k-1}^{-2}[\tfrac{1}{2}(k-1)(2k-3)]. \tag{2.15}$$

The value of λ will depend upon the forms of f and g. In the next section the quantity $(f^2-1)/g^2$ in (2.14) will appear as a Casimir operator of a dynamical potential algebra, so that its constancy appears naturally in the algebraic approach.

Equation (2.15) is invalid if $k=1$ or $\frac{3}{2}$. Treating these cases separately yields

$$N_1^{-2} = [\tfrac{1}{2}h^2]_\alpha^\infty \tag{2.16}$$

$$N_{3/2}^{-2} = (4-\lambda)^{-1}[(2g+f)h^2]_\alpha^\infty. \tag{2.17}$$

The physical interpretation of these results has been indicated above, but we should observe that this does depend on the solutions being normalizable. This question will be deferred to section 6, as will the matter of whether (2.13) is zero, because one requires explicit forms for $f(x)$ and $g(x)$. At this stage we remark that applying J_+ may eventually produce an unphysical state; i.e. ψ_n becomes non-normalizable for sufficiently large n. Note that J_+ will not give zero, because the above formula for $J_+|km\rangle$ then requires either $m=k-1$ or $m=-k$.

A final point regarding the potential algebra is that we do not assume that the values of k and m are restricted to integers and half-integers. We are thus using projective representations of the corresponding group (Alhassid *et al* 1986); in a linear representation of the algebra the eigenvalues of J_0 all have the same fractional part, which is therefore an invariant of the representation (Barut and Fronsdal 1965).

3. Dynamical potential algebras

A representation space of a dynamical potential algebra contains wavefunctions corresponding to both different potentials and different energies. The operator realizations of the algebra elements will change the energy and the Hamiltonian. Since the potential algebra is usually a subalgebra, we proceed to obtain dynamical potential algebras by enlarging the above so(2, 1) algebra. The method is to first obtain iso(2, 1) using three commuting operators which are components of a vector $\boldsymbol{P}$ (Ui 1968) with respect to the so(2, 1) transformations, and then to obtain so(2, 2), for example, by replacing $\boldsymbol{P}$ by $\frac{1}{2}\mathrm{i}[\boldsymbol{P}, J^2]$. This expansion of the algebra is the reverse of contraction (Gilmore 1974).

For any real functions $F(x)$ and $G(x)$, the multiplicative operators

$$P_\pm = \mathrm{e}^{\pm \mathrm{i}\phi} G(x) \qquad P_0 = F(x)$$

commute, $P_\pm$ are Hermitian conjugates and $[J_0, P_0]=0$, $[J_0, P_\pm]=\pm P_\pm$. The remaining iso(2, 1) commutation relations

$$[J_\pm, P_0] = \mp P_\pm \qquad [J_\pm, P_\pm]=0 \qquad [J_\pm, P_\mp] = \mp 2P_0 \tag{3.1}$$

are satisfied if

$$F' = -G \qquad G' = fG = -fG - 2F. \tag{3.2}$$

Equations (3.1) or (3.2) are linear and homogeneous in F and G, which therefore contain an arbitrary multiplicative constant. From (2.2), $G'=fG$ becomes $G'/G = -g'/g$ and $G=1/g$ (leaving the constant to be included later when necessary), and then $F=-fG=-f/g$. From (2.2) the remaining equation $F'=-G$ is satisfied, and the required operators are (any real multiple of)

$$P_\pm = \mathrm{e}^{\pm \mathrm{i}\phi}/g \qquad P_0 = -f/g. \tag{3.3}$$

The Casimir operators of this iso(2, 1) algebra are $\boldsymbol{P}\cdot\boldsymbol{J} = -1$, and $\boldsymbol{P}^2 = (f^2-1)/g^2$. In any representation we have (2.14), and the constant λ in (2.15) is just the value of $\boldsymbol{P}^2$.

Expansion of the iso(2, 1) algebra involves replacing $\boldsymbol{P}$ by

$$\boldsymbol{Y} = \tfrac{1}{2}\mathrm{i}[\boldsymbol{P}, J^2] \tag{3.4}$$

where i is included so that

$$Y_0^\dagger = Y_0 \qquad Y_-^\dagger = Y_+. \tag{3.5}$$

Using (3.1) gives the following expressions for the components of $\boldsymbol{Y}$:

$$\begin{aligned} Y_0 &= \tfrac{1}{2}\mathrm{i}(P_-J_+ - P_+J_-) - \mathrm{i}P_0 \\ Y_\pm &= \pm\mathrm{i}(P_0J_\pm - P_\pm J_0) - \mathrm{i}P_\pm . \end{aligned} \tag{3.6}$$

Then

$$[J_0, Y_0] = [J_\pm, Y_\pm] = 0 \qquad [J_\pm, Y_\mp] = \mp 2 Y_0$$
$$[J_0, Y_\pm] = [Y_0, J_\pm] = \pm Y_\pm \tag{3.7}$$

(i.e. $\boldsymbol{Y}$ is a vector), and

$$[Y_0, Y_\pm] = \mp\lambda J_\pm \qquad [Y_+, Y_-] = 2\lambda J_0 \tag{3.8}$$

where $\lambda = P^2$. Since $[P^2, \boldsymbol{Y}] = \boldsymbol{0}$, we can assume the value of P^2 is constant in an irreducible representation of the expanded algebra.

Equations (2.3), (3.8) and (3.7) are standard commutation relations of:

(i) so(2, 2) if $\lambda = -1$,

(ii) iso(2, 1) if $\lambda = 0$ or

(iii) so(3, 1) if $\lambda = 1$.

For any other non-zero value of λ, the so(2, 2) or so(3, 1) commutation relations are exhibited by taking $\boldsymbol{Q} = \boldsymbol{Y}|\lambda|^{-1/2}$ in the basis of the algebra, giving so(2, 2) when $\lambda < 0$ and so(3, 1) when $\lambda > 0$.

The so(2, 2) and so(3, 1) algebras have the Casimir operators $Q_0J_0 - \frac{1}{2}Q_+J_- - \frac{1}{2}Q_-J_+ = \boldsymbol{Q}\cdot\boldsymbol{J} = \boldsymbol{J}\cdot\boldsymbol{Q}$ and $J^2 \pm Q^2 = J^2 - Y^2/\lambda$. For the iso(2, 1) algebra the Casimir operators are $\boldsymbol{Y}\cdot\boldsymbol{J} = \boldsymbol{J}\cdot\boldsymbol{Y}$ and Y^2. From (3.6) one can show that

$$Y^2 = -(\boldsymbol{P}\cdot\boldsymbol{J})^2 + P^2(J^2+1). \tag{3.9}$$

Finally, let us write $\boldsymbol{Y}$ and Y^2 in terms of the functions f and g that satisfy (2.2):

$$Y_0 = \frac{\mathrm{i}}{g}\left(\frac{\partial}{\partial x} + \tfrac{1}{2}f\right)$$
$$Y_\pm = \frac{\mathrm{e}^{\pm\mathrm{i}\phi}}{g}\left(-\mathrm{i}f\frac{\partial}{\partial x} \mp f'\frac{\partial}{\partial\phi} - \tfrac{1}{2}\mathrm{i}f' - \tfrac{1}{2}\mathrm{i} \pm \mathrm{i}g'\right) \tag{3.10}$$

$$Y^2 = -1 + \lambda(J^2+1) \tag{3.11}$$

where J^2 is given in equation (2.4).

4. Representations of the algebras

In the previous section we have obtained operators $\boldsymbol{J}$ and $\boldsymbol{Q}$ which realize a basis of so(2, 2) or of so(3, 1), and operators $\boldsymbol{J}$ and $\boldsymbol{Y}$ realizing a basis of iso(2, 1). We now consider the representations of these algebras obtained with the representation spaces defined in (2.5). Specification of a representation includes giving the values of the Casimir operators, a basis of the representation space, and the action of the algebra operators in this basis.

The first Casimir operator ($\boldsymbol{Q}\cdot\boldsymbol{J}$ or $\boldsymbol{Y}\cdot\boldsymbol{J}$) has the value zero. This follows directly from (3.6). Using (3.9) shows that the second Casimir operator for so(2, 2) and so(3, 1) is $-1 + (\boldsymbol{P}\cdot\boldsymbol{J})^2/\lambda$ with the value $(-1+\lambda^{-1})$. A natural description of so(2, 2) representations (Wu *et al* 1987, 1989) uses coordinates on a four-dimensional hyperboloid and writes the second Casimir eigenvalue as $\omega(\omega+2)$; we may identify ω with $-1 + \mathrm{i}|\lambda|^{-1/2}$.

To obtain the action of the algebra operators on the basis functions $|k\, m\rangle$ considered in section 2, we first consider the action of the operators $\boldsymbol{P}$. Let $P_{\pm 1} = 2^{-1/2}P_\pm$, so that $P_\mu(\mu = 0, \pm 1)$ are the 'circular' components of the so(2, 1) vector $\boldsymbol{P}$. Then taking the products $P_\mu|k\,m\rangle$ amounts to coupling the three-dimensional (non-unitary) representation of so(2, 1) with the positive discrete series representation D_k^+. We therefore expect an equation of the form

$$P_\mu|k\,m\rangle=\sum_{j=k-1}^{k+1}\langle j\,\|\boldsymbol{P}\|\,k\rangle\langle k\,m,1\mu|j\,m+\mu\rangle|j\,m+\mu\rangle \tag{4.1}$$

where the $\langle k\,m,1\mu|j\,m+\mu\rangle$ are so (2, 1) Wigner coefficients. Explicit expressions for these Wigner coefficients (Ui 1968) are shown in table 1. Using these results evidently assumes that $k>1$. Then (4.1) gives $P_{-1}|k\,k\rangle=\langle k-1\,\|\boldsymbol{P}\|\,k\rangle|k-1\;k-1\rangle$, which can be compared with an explicit evaluation using (2.9), (2.15) and (3.3):

$$(\mathrm{e}^{-\mathrm{i}\phi}/g\sqrt{2})N_k g^{k-1/2}h\,\mathrm{e}^{\mathrm{i}k\phi}=\langle k-1\,\|\boldsymbol{P}\|\,k\rangle N_{k-1}g^{k-3/2}h\,\mathrm{e}^{\mathrm{i}(k-1)\phi}.$$

Evidently, provided $k>\frac{3}{2}$,

$$\langle k-1\,\|\boldsymbol{P}\|\,k\rangle=N_k/N_{k-1}\sqrt{2}=\left(\frac{1-\lambda(k-1)^2}{(k-1)(2k-3)}\right)^{1/2}. \tag{4.2}$$

Similarly (4.1) gives

$$P_0|k\,k\rangle=-\langle k\,\|\boldsymbol{P}\|\,k\rangle\left(\frac{k}{k-1}\right)^{1/2}|k\,k\rangle+\langle k-1\,\|\boldsymbol{P}\|\,k\rangle\left(\frac{1}{k-1}\right)^{1/2}|k-1\,k\rangle.$$

Substituting (2.9), (3.3) and (2.10), and using (2.15) and (4.2), leads to

$$\langle k\,\|\boldsymbol{P}\|\,k\rangle=(k^2-k)^{-1/2}. \tag{4.3}$$

Note that (2.10) is used with k replaced by $k-1$, and with the normalization factor $(2k-2)^{-1/2}$.

Finally, because $P_{\pm1}$ are Hermitian conjugates, (4.1) implies

$$\langle k+1\,\|\boldsymbol{P}\|\,k\rangle=\langle k\,\|\boldsymbol{P}\|\,k+1\rangle\left(\frac{2k-1}{2k+1}\right)^{1/2} \tag{4.4}$$

$$=\left(\frac{1-\lambda k^2}{k(2k+1)}\right)^{1/2}. \tag{4.5}$$

These results assume that the functions $|k\,m\rangle$ can be normalized and that the matrix elements of $\boldsymbol{P}$ exist. The condition $k>\frac{3}{2}$ is related to these requirements, which will be discussed further in section 6.

Table 1. Expressions for so(2, 1) Wigner coefficients $\langle k\,m,1\mu|j\,m+\mu\rangle$.

	$\mu=1$	$\mu=0$	$\mu=-1$
$j=k+1$	$\left(\frac{(m+k)(m+k+1)}{2k(2k-1)}\right)^{1/2}$	$\left(\frac{(m-k)(m+k)}{k(2k-1)}\right)^{1/2}$	$\left(\frac{(m-k)(m-k-1)}{2k(2k-1)}\right)^{1/2}$
$j=k$	$-\left(\frac{(m+k)(m-k+1)}{2k(k-1)}\right)^{1/2}$	$-\left(\frac{m^2}{k(k-1)}\right)^{1/2}$	$-\left(\frac{(m-k)(m+k-1)}{2k(k-1)}\right)^{1/2}$
$j=k-1$	$\left(\frac{(m-k+1)(m-k+2)}{(2k-1)(2k-2)}\right)^{1/2}$	$\left(\frac{(m+k-1)(m-k+1)}{(2k-1)(k-1)}\right)^{1/2}$	$\left(\frac{(m+k-1)(m+k-2)}{(2k-1)(2k-2)}\right)^{1/2}$

The action of Y on the basis functions is also given by an equation like (4.1), and the definition (3.4) directly gives

$$(2k-1)^{1/2}\langle k \,\|Y\|\, k+1\rangle = -(2k+1)^{1/2}\langle k+1 \,\|Y\|\, k\rangle = \mathrm{i}(k-\lambda k^3)^{1/2} \tag{4.6}$$

and

$$\langle k \,\|Y\|\, k\rangle = 0. \tag{4.7}$$

Including a factor $|\lambda|^{-1/2}$ gives the matrix elements of $\boldsymbol{Q}$.

5. Matrix elements connecting eigenstates in the same potential

We observed in section 2 that the functions written down in (2.9), (2.10) and (2.11) were eigenfunctions corresponding to different potentials V_k, V_{k+1} and V_{k+2}. Throughout this section, as before, we will assume the functions used are normalizable, and will remark on any related conditions that appear during the work. Eigenfunctions in the same potential V_M, given in (2.7), are obtained by taking $k=M$, $k=M-1$ and $k=M-2$ in (2.9), (2.10) and (2.11) respectively. If we denote (2.9) by $\psi_0(k)$, these eigenfunctions are

$$\mathrm{e}^{-\mathrm{i}M\phi} J_+^n \psi_0(M-n)\,\mathrm{e}^{\mathrm{i}(M-n)\phi} \qquad (n=0,1,2,\ldots) \tag{5.1}$$

belonging to the eigenvalues $-(M-n-\frac{1}{2})^2$. When these functions are constructed, normalization may be preserved, as discussed in section 2. The first three normalized eigenfunctions are

$$\chi_1 = N_M g^{M-1/2} h \tag{5.2}$$

$$\chi_2 = N_M\left(\frac{(2M-3)}{[1-\lambda(M-1)^2]}\right)^{1/2}(g-Mf+f)g^{M-3/2}h \tag{5.3}$$

$$\chi_3 = N_M\left(\frac{(M-1)(M-\frac{5}{2})}{2[1-\lambda(M-1)^2][1-\lambda(M-2)^2]}\right)^{1/2}$$
$$\times[2g^2-2(2M-3)fg+2(M-2)(M-1)f^2-M+2]g^{M-5/2}h. \tag{5.4}$$

The factors appearing in the normalization constants indicate that χ_2 is not an eigenfunction (not normalizable) if $M \leqslant \frac{3}{2}$, and similarly (5.4) requires $M > \frac{5}{2}$. These conditions will be derived in the next section, using the asymptotic behaviour of f, g, and h.

In this section we consider matrix elements

$$\langle l\,|R|\,l'\rangle = \int_\alpha^\infty \chi_l(R\chi_{l'})\,\mathrm{d}x \tag{5.5}$$

where R is some operator defined on functions of x with domain (α, ∞). The work in the previous section then gives

$$\langle l|(-f/g)|l\rangle = \langle M-l+1\ M|P_0|M-l+1\ M\rangle \tag{5.6}$$

$$\langle l|(-f/g)|l-1\rangle = \langle M-l+1\ M|P_0|M-l+2\ M\rangle \tag{5.7}$$

and, similarly, the matrix elements of Y_0 are given in (3.6). Other non-zero matrix elements given before are inapplicable to (5.5), which refers to eigenfunctions in the same potential.

An alternative approach (López Pineiro and Moreno 1988) is to use hypervirial theorems, which apply to the matrix elements of any operator which can be expressed as a commutator involving the Hamiltonian H. An example is given by the definition of Y_0, since J^2 can be replaced by $(-H-\frac{1}{4})$:

$$Y_0 = \tfrac{1}{2}\mathrm{i}HP_0 - \tfrac{1}{2}\mathrm{i}P_0H. \tag{5.8}$$

Then, putting $E_l = -(M-l+\frac{1}{2})^2$,

$$\langle l|Y_0|l'\rangle = \tfrac{1}{2}\mathrm{i}(E_l - E_{l'})\langle l|P_0|l'\rangle. \tag{5.9}$$

This gives (4.7), and the ratios of the matrix elements in (4.6) and either (4.2) or (4.4). Also (3.6) give

$$Y_0J^2 - J^2Y_0 = \mathrm{i}P_0J^2 + \mathrm{i}J^2P_0 - 2\mathrm{i}(\boldsymbol{P}\cdot\boldsymbol{J})J_0. \tag{5.10}$$

Hence, as $\boldsymbol{P}\cdot\boldsymbol{J} = -1$ and $J_0 = M$,

$$(E_l - E_{l'})\langle l|Y_0|l'\rangle = -\mathrm{i}(E_l + E_{l'} + \tfrac{1}{2})\langle l|P_0|l'\rangle + 2\mathrm{i}M\delta_{l,l'} \tag{5.11}$$

which agrees with (4.1) and (4.3) when $l = l'$ (putting $k = M - l + 1$). For $l \neq l'$, the condition for a non-trivial solution of (5.9) and (5.11) is

$$(E_l - E_{l'})^2 + 2E_l + 2E_{l'} + 1 = 0$$

which may be written

$$(E_l - E_{l'} + 1)^2 = -4E_{l'} = (2M - 2l' + 1)^2.$$

This reduces to

$$(M - l' + \tfrac{1}{2} \mp 1)^2 = (M - l + \tfrac{1}{2})^2$$

finally giving $l' \pm 1 = l$ as the condition for non-zero off-diagonal matrix elements of Y_0 and P_0.

From (5.6) and (5.7), the non-zero values are

$$\langle l|(-f/g)|l\rangle = -M/(M-l)(M-l+1) \tag{5.12}$$

$$\langle l|(-f/g)|l-1\rangle = \left(\frac{[1-\lambda(M-l+1)^2](2M-2l+1)(l-1)}{(M-l+1)^2(2M-2l+1)(2M-2l+3)}\right)^{1/2} \tag{5.13}$$

$$\langle l|\frac{\mathrm{i}}{g}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \tfrac{1}{2}f\right)|l-1\rangle = \mathrm{i}(M-l+1)\langle l|(-f/g)|l-1\rangle. \tag{5.14}$$

In calculating (5.12) a factor $(M-l)^{1/2}$ is squared, suggesting that the condition $l < M$ may be required.

6. Eigenvalues and eigenfunctions for Gendenshtein and Morse potentials

In this section the functions f and g will be given explicitly, and results thus obtained for specific potentials. There are three classes of solutions of (2.2) according to whether $f^2 < 1$, $f^2 = 1$ or $f^2 > 1$:

(I) $\quad f(x) = \tanh(x-c) \qquad g(x) = b\,\mathrm{sech}(x-c) \qquad$ (6.1)

(II) $\quad f(x) = \pm 1 \qquad g(x) = b\,\mathrm{e}^{\mp x} \qquad$ (6.2)

(III) $\quad f(x) = \coth(x-c) \qquad g(x) = b\,\mathrm{cosech}(x-c). \qquad$ (6.3)

Since the Schrödinger equation (2.6) will be solved on an unbounded interval, results of physical significance will be independent of c, and we therefore take $c = 0$. Similarly it is sufficient to take only the upper sign in (6.2). Substituting into (2.7) gives three classes of potentials:

(I) $$V_1(x) = (b^2 - M^2 + \tfrac{1}{4}) \operatorname{sech}^2 x - 2Mb \operatorname{sech} x \tanh x. \tag{6.4}$$

This potential was discovered by Gendenshtein (1983) while finding potentials allowing an algebraic solution of the eigenvalue problem. The wavefunctions have also been given (Dabrowska *et al* 1988, Pertsch 1990). For $b = 0$, $V_1(x)$ reduces to the Rosen-Morse potential, but most of our work is invalid when $g(x) = 0$. However, the potential algebra is valid, so we explain the observation (Pertsch 1990) that the Rosen–Morse and Gendenshtein potentials have the same eigenvalues.

(II) After a translation $x \to (x - \log M)$,

$$V_2(x) = M^2 b^2 \,\mathrm{e}^{-2x} - 2M^2 b\,\mathrm{e}^{-x} \tag{6.5}$$

which is a Morse potential. Potentials V_1 and V_2 are considered on the interval $(-\infty, \infty)$.

(III) $$V_3(x) = (b^2 + M^2 - \tfrac{1}{4}) \operatorname{cosech}^2 x - 2Mb \coth x \operatorname{cosech} x. \tag{6.6}$$

This potential, also obtained by Gendenshtein, is singular at $x = 0$. The Schrödinger equation is therefore solved on $(0, \infty)$ and we can also interpret the results in terms of the s-wave of a spherically symmetric problem with potential $V_3(r)$. The singularity is attractive if $|b - M| < \frac{1}{2}$ and repulsive if $|b - M| > \frac{1}{2}$. For $b = M \pm \frac{1}{2}$ one gets the non-singular Rosen–Morse potentials

$$V(x) = -\tfrac{1}{2}(2b^2 \mp b) \operatorname{sech}^2\left(\frac{x}{2}\right). \tag{6.7}$$

In case I we can assume $b \geq 0$, because $(x, b) \to (-x, -b)$ leaves (6.4) invariant; in cases II and III we can assume $b > 0$ since otherwise (6.5) and (6.6) are non-negative.

Wu and Alhassid (1990) only considered (6.1) and (6.3) with $b = 0$, and so did not discuss the Gendenshtein potentials.

Some of the potentials are illustrated in figures 1–4.

To consider whether solutions are normalizable, we first give in table 2 the asymptotic behaviour of f, g, and $h = \exp(\int g(x)\,\mathrm{d}x)$. The values of h are ambiguous to the extent of a multiplicative constant, but this can be absorbed into the normalization constant N_M, and does not affect whether a function is normalizable.

All three cases have the same qualitative behaviour as $x \to \infty$: the normalization integral will converge provided all powers of g in the function are positive. Because $g' = -fg$, applying J_+ does not decrease the lowest power of g, as in (2.9)–(2.11). Thus in (5.1) the lowest power of g is $M - n - \frac{1}{2}$, determined by $\psi_0(M - n)$, and so we require $n < M - \frac{1}{2}$.

In case I, the same condition applies as $x \to -\infty$, while in case II normalizability is ensured by the form of $h(-\infty)$. Hence for potentials (6.4) and (6.5), there are n_1 eigenvalues,

$$-(M - \tfrac{1}{2})^2, -(M - \tfrac{3}{2})^2, \ldots, -(M - n_1 + \tfrac{1}{2})^2 \tag{6.8}$$

where n_1 is determined by

$$n_1 - \tfrac{1}{2} < M \leq n_1 + \tfrac{1}{2}. \tag{6.9}$$

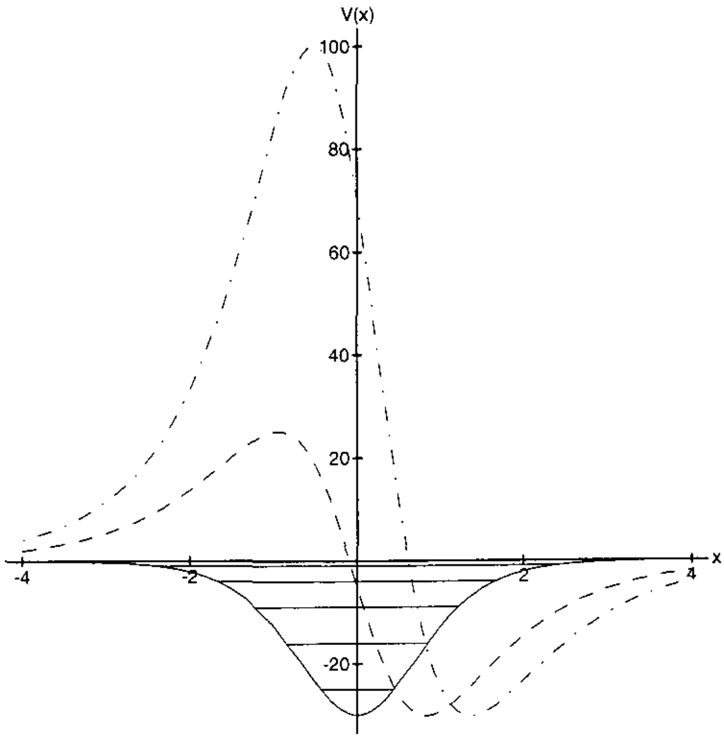

Figure 1. Examples of potentials $V_t(x)$ with the same spectrum. The parameters are $M = 5.5$ and $b = 0, 5, 10$ for the full, broken, and chain lines respectively. The common bound state spectrum is plotted inside the Rosen–Morse potential well corresponding to $b = 0$.

Since these conditions are independent of b, they include the Rosen–Morse potential obtained from (6.4) when $b = 0$.

In case III, the behaviour as $x \to \alpha = 0$ involves f, g and h, and is determined by the highest powers of f and g. From (2.1) and (2.2), applying J_+ increases by 1 the total power of f and g, as in equations (2.9)–(2.11). In (5.1) the power is always $M - \frac{1}{2}$, and including h gives the behaviour $x^{b-M+1/2}$ as $x \to 0$. We will therefore require $M < b + \frac{1}{2}$, making the eigenfunction zero at $x = 0$ (a stronger condition than the existence of the normalization integral). The eigenvalues are still given by (6.8) and (6.9), provided $M < b + \frac{1}{2}$. If $M > b + \frac{1}{2}$, the solutions have an unacceptable singularity at $x = 0$. We should also mention the two special cases (6.7), where the eigenfunctions are even or odd since $V(x)$ is even. If $b = M + \frac{1}{2}$, the solutions are zero at $x = 0$, and (6.8) gives the eigenvalues corresponding to odd eigenfunctions only. If $b = M - \frac{1}{2}$, the solutions are finite but non-zero at $x = 0$, and (6.8) gives the eigenvalues corresponding to even eigenfunctions only. These statements agree with the Rosen–Morse eigenvalues obtained in case I (after allowing for the $x/2$ in (6.7) compared to the x in (6.4)). In general the first application of J_+ that makes $m > b + \frac{1}{2}$ produces a function which is singular at $x = 0$.

In all cases the eigenfunctions are given by (5.1), and the normalized eigenfunctions involve a single constant N_M, as in (5.2)–(5.4). The constants satisfy (2.15), in which we can now express the value of λ in terms of the potential constants, and verify that condition (2.13) requires $k > \frac{3}{2}$ in all cases, and in addition $k < b + 1$ in case III. When

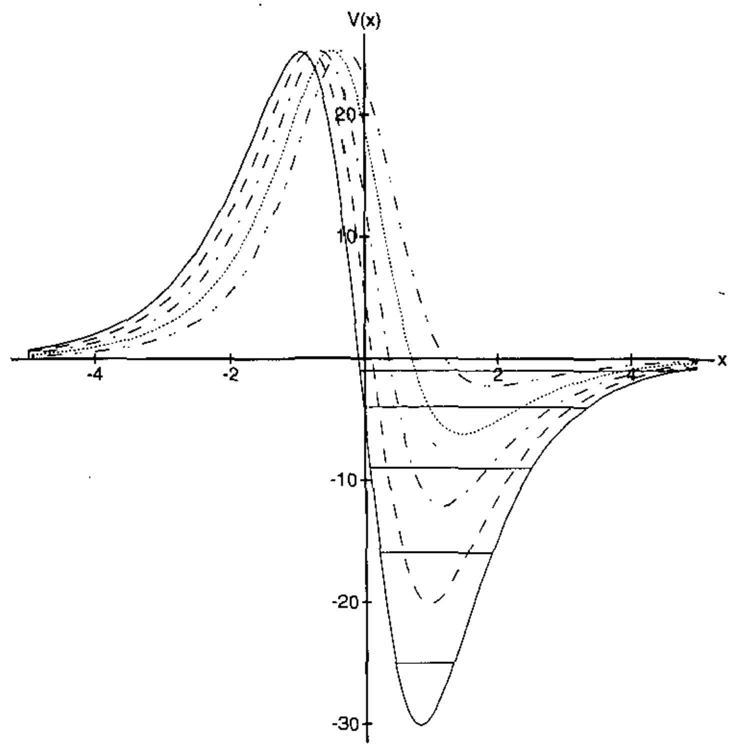

Figure 2. Examples of potentials $V_1(x)$ with 5, 4, 3, 2 or 1 bound state (full, broken, chain, dotted, and double-dotted chain lines respectively). The corresponding parameters are $b=5$ and $M=5.5, 4.5, 3.5, 2.5$ and 1.5. The five-level spectrum of the first potential is plotted.

M is an integer or half-integer, (2.16) or (2.17) allow the constant to be determined completely. The results are shown in table 3. These values are obtained using table 2, and correspond to the following functions h in the three cases:

$$h(x)=\exp[b\tan^{-1}(\sinh x)] \tag{6.10I}$$

$$h(x)=\exp(-b\,\mathrm{e}^{-x}) \tag{6.10II}$$

$$h(x)=(\tanh\tfrac{1}{2}x)^b. \tag{6.10II}$$

(The formulae given for N_M are actually valid for any real $M>1$ if $(2M-2)!$ is replaced by $\Gamma(2M-1)$; however, we are unable to show this by the algebraic methods of this paper.)

In the previous sections formulae were given for the matrix elements of P_0 and Y_0. We can now identify these operators for the three cases (see table 4). In case II the appearance of M is due to the translation $x\to(x-\log M)$ used to obtain (6.5).

The existence of the matrix elements of these operators also depends (as for the normalization conditions) on the asymptotic forms given in table 2. As $x\to\infty$ the convergence of a matrix element integral is determined in all three cases by the factor $(1/g)$ in the operators. This gives the (previously conjectured) condition $l<M$ for the diagonal elements. Considering $x\to\alpha$ ($\alpha=-\infty$ or 0) shows that no further conditions are required. In case III, as $x\to 0$, powers of x are unchanged from the normalization integral since $(f/g)\to\lambda^{1/2}=1/b$, and $(1/g)\to(x/b)$ so that $g^{-1}(\mathrm{d}/\mathrm{d}x)$ does not change the power.

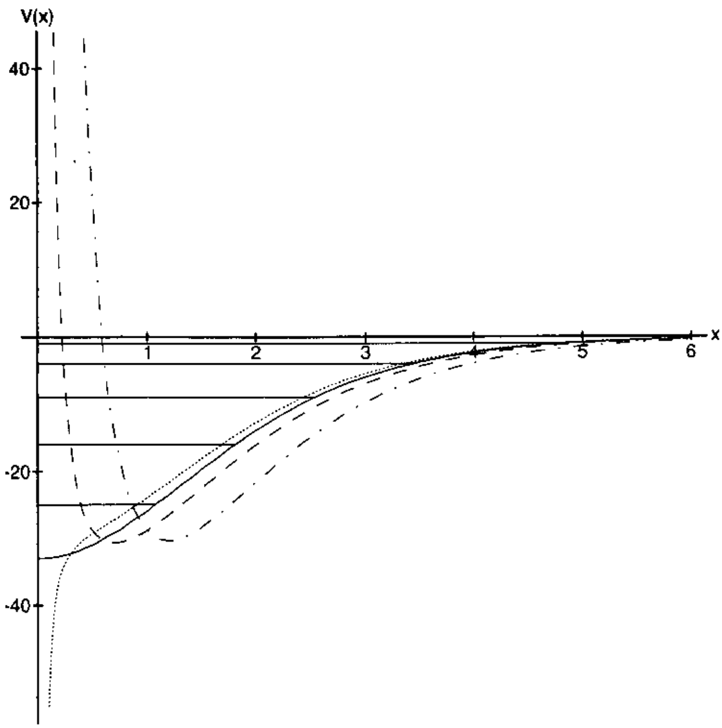

Figure 3. Examples of potentials $V_3(x)$ with the same spectrum. The parameters are $M = 5.5$ and $b = 5.5$, 6, 7 and 10 for the dotted, full, broken and chain lines respectively. The common bound state spectrum is plotted inside the right half of the Rosen–Morse potential well, corresponding to $b = 6$. As explained in the text, for the latter it consists of the odd eigenstates of the symmetrical Rosen–Morse well.

The condition $l < M$ is violated for the highest energy state ($l = n_1$) when either M is an integer or the fractional part of M exceeds $\frac{1}{2}$.

7. Ladder operators connecting eigenstates in the same potential

The purpose of this section is to obtain some operators $X_\pm$ which are ladder operators for the eigenfunctions χ_l: $X_\pm\chi_l = c_\pm\chi_{l\pm1}$. Such operators are known for some special cases of the potential (2.7) considered here: the Morse potential (Nieto and Simmons 1979, Berrondo and Palma 1980, Levine 1982), the Rosen–Morse potential (Englefield 1987) and the Gendenshtein potential (Fellemans 1989). Their ladder operators require the eigenfunctions to be multiplied by a phase factor $e^{i\kappa\gamma}$, where γ is an introduced non-physical variable. The constant κ, analogous to M in the potential algebra, is related to the energy eigenvalue and changes by 1 when the ladder operators are applied. The following work essentially sets the operators used by Fellemans into our more general context.

Since the potential will not change, the factors $e^{iM\phi}$ will now be discarded, and in all previous operators $-i\partial/\partial\phi$ replaced by M. The eigenfunctions χ_l belonging to the energy eigenvalue $-(M-l+\frac{1}{2})^2$ will be multiplied by $e^{i\gamma(M-l+1/2)}$, so that $\partial^2/\partial\gamma^2$ is equivalent to H.

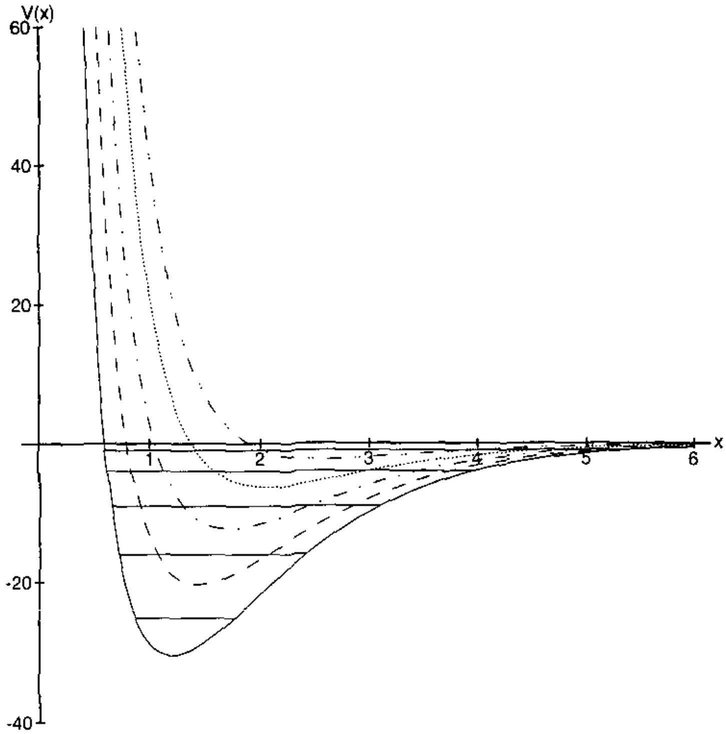

Figure 4. Examples of potentials $V_3(x)$ with 5, 4, 3, 2 or 1 bound state (full, broken, chain, dotted and double-dotted chain lines respectively). The corresponding parameters are $b=10$ and $M=5.5$, 4.5, 3.5, 2.5 and 1.5. The five-level spectrum of the first potential is plotted.

Table 2. Asymptotic behaviour of $f(x)$, $g(x)$ and $h(x)$.

	i ($\alpha=-\infty$)	ii ($\alpha=-\infty$)	iii ($\alpha=0$)
$f(\alpha)$	-1	1	$(1/x)\to\infty$
$f(\infty)$	1	1	1
$g(\alpha)$	$2b\,\mathrm{e}^{x}\to 0^+$	$b\,\mathrm{e}^{-x}\to\infty$	$b/x\to+\infty$
$g(\infty)$	$2b\,\mathrm{e}^{-x}\to 0^+$	$b\,\mathrm{e}^{-x}\to 0^+$	$2b\,\mathrm{e}^{-x}\to 0^+$
$h(\alpha)$	$\mathrm{e}^{-b\pi/2}$	$\exp(-b\,\mathrm{e}^{-x})\to 0^+$	$x^b\to 0^+$
$h(\infty)$	$\mathrm{e}^{b\pi/2}$	1	1

From the two operators P_0 and Y_0 connecting an eigenfunction χ_l with the eigenfunctions χ_{l+1}, χ_{l-1} corresponding to the same potential, one can construct two linear combinations $\alpha_\pm(l)P_0+\beta_\pm(l)Y_0$ such that

$$\langle l\pm1|(\alpha_\pm(l)P_0+\beta_\pm(l)Y_0)|l\rangle=0. \tag{7.1}$$

The results of sections 4 and 5 show that $\alpha_\pm(l)=\mp\mathrm{i}(M-l+\frac{1}{2}\mp\frac{1}{2})\beta_\pm(l)$, where we shall choose $\beta_\pm(l)=\pm1$. Since P_0 has non-vanishing diagonal matrix elements, the same is true for these two linear combinations, and one finds

$$\langle l|(\alpha_\pm(l)P_0+\beta_\pm(l)Y_0)|l\rangle=\mathrm{i}M(M-l+\tfrac{1}{2}\pm\tfrac{1}{2})^{-1}. \tag{7.2}$$

Table 3. Ground state normalization constants.

	Case I	Case II	Case III
λ	$-1/b^2$	0	$1/b^2$
N_1	$(\sinh b\pi)^{-1/2}$	$\sqrt{2}$	$\sqrt{2}$
$N_{3/2}$	$\left(\dfrac{4b^2+1}{2b^2\cosh b\pi}\right)^{-1/2}$	2	$(4-1/b^2)^{1/2}$ if $b>\frac{1}{2}$
N_L (L integer)	$\left(\dfrac{2}{b}\right)^{L-1}\dfrac{\lvert\Gamma(L+\mathrm{i}b)\rvert}{[b\pi(2L-2)!]^{1/2}}$	$\dfrac{2^L}{[2(2L-2)!]^{1/2}}$	$\left(\dfrac{2}{b}\right)^{L-1/2}\left(\dfrac{\Gamma(b+L)}{\Gamma(b-L+1)(2L-2)!}\right)^{1/2}$
$N_{L+1/2}$ (L integer)	$\dfrac{2^L\lvert\Gamma(L+\frac{1}{2}+\mathrm{i}b)\rvert}{b^L[2\pi(2L-1)!]^{1/2}}$	$\dfrac{2^L}{[(2L-1)!]^{1/2}}$	$\dfrac{2^L}{b^L}\left(\dfrac{\Gamma(b+L+\frac{1}{2})}{\Gamma(b-L+\frac{1}{2})(2L-1)!}\right)^{1/2}$

Table 4. Operators with known matrix elements.

	Case I	Case II	Case III
Potential	(6.4)	(6.5)	(6.6)
$-bP_0=bf/g$	$\sinh x$	e^x/M	$\cosh x$
$-\mathrm{i}bY_0=\dfrac{b}{g}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}+\dfrac{f}{2}\right)$	$(\cosh x)\dfrac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\sinh x$	$\left(e^x\dfrac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}e^x\right)\Big/M$	$\sinh x\dfrac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\cosh x$

After multiplying the eigenfunctions χ_l by $e^{\mathrm{i}\gamma(M-l+1/2)}$, $M-l+\frac{1}{2}$ may be replaced by the operator

$$L_0=-\mathrm{i}\frac{\partial}{\partial\gamma}. \tag{7.3}$$

The two linear combinations (multiplied by $e^{\pm\mathrm{i}\gamma}$) become the operators

$$K_\pm=e^{\pm\mathrm{i}\gamma}\left(\pm Y_0\pm\tfrac{1}{2}\mathrm{i}P_0-P_0\frac{\partial}{\partial\gamma}\right)=e^{\pm\mathrm{i}\gamma}\,g^{-1}\left(\pm\mathrm{i}\frac{\partial}{\partial x}+f\frac{\partial}{\partial\gamma}\right) \tag{7.4}$$

with vanishing matrix elements between $\chi_l\,e^{\mathrm{i}\gamma(M-l+\frac{1}{2})}$ and $\chi_{l\pm1}\,e^{\mathrm{i}\gamma(M-l+1/2\mp1)}$ respectively.

The three operators L_0, K_+, K_- close under commutation as follows:

$$[L_0,K_\pm]=\pm K_\pm \qquad [K_+,K_-]=2\lambda L_0 \tag{7.5}$$

where as before $\lambda=\boldsymbol{P}^2$. Since K_+ and K_- are not adjoint, they span with L_0 a complex Lie algebra. For $\lambda=0$, (7.5) are standard commutation relations of iso(2), whereas for $\lambda\neq0$, they become standard commutation relations of so(3),

$$[L_0,L_\pm]=\pm L_\pm \qquad [L_+,L_-]=2L_0 \tag{7.6}$$

after replacing $K_\pm$ by

$$L_\pm=\lambda^{-1/2}K_\pm \tag{7.7}$$

where $\lambda^{-1/2}$ is complex for $\lambda<0$.

The iso(2) and so(3) algebras have the Casimir operators

$$K^2=K_+K_-=\frac{1}{g^2}\left(-L_0^2+\frac{\partial^2}{\partial x^2}\right) \tag{7.8a}$$

and

$$L^2 = L_+L_- + L_0^2 - L_0 = \frac{1}{\lambda g^2}\left(-L_0^2 + \frac{\partial^2}{\partial x^2}\right) \tag{7.8b}$$

respectively. Then from (7.8) and (2.7) we can write

$$H = -\frac{\partial^2}{\partial x^2} + V_M = -L_0^2 - g^2 W \tag{7.9}$$

where

$$W = \frac{2Mf}{g} - 1 + K^2 \qquad \text{if } \lambda = 0 \tag{7.10a}$$

$$W = \frac{2Mf}{g} - 1 + \lambda(L^2 - M^2 + \tfrac{1}{4}) \qquad \text{if } \lambda \neq 0. \tag{7.10b}$$

It may be verified that W gives zero when operating on any eigenfunction. In other words, in the space spanned by the bound states, the operators $(K^2 + 2Mf/g)$ and $(L^2 + 2Mf/\lambda g)$ have the constant values 1 and $(\lambda^{-1} + M^2 - \frac{1}{4})$, respectively.

From (7.2), the ladder operators for raising and lowering the energy are given by

$$X_\pm = -2M\,\mathrm{e}^{\mp\mathrm{i}\gamma} - \mathrm{i}\,K_\mp(2L_0 \mp 1). \tag{7.11}$$

Their eigenvalue shift property can also be directly verified by noting that since H commutes with $\mathrm{e}^{\pm\mathrm{i}\gamma}$ and L_0, and

$$[K_\mp, F(x)] = \mp\mathrm{i}\,\mathrm{e}^{\mp\mathrm{i}\gamma} F'(x)/g(x) \tag{7.12}$$

one may write

$$\begin{aligned}[X_\pm, H] &= -\mathrm{i}[K_\mp, H](2L_0 \mp 1)\\ &= \mp X_\pm(2L_0 \mp 1) \mp Z.\end{aligned} \tag{7.13}$$

Here

$$Z = 2\,\mathrm{e}^{\mp\mathrm{i}\gamma} fg(2L_0 \mp 1)\,W \tag{7.14}$$

gives zero when acting on any eigenfunction. Applying (7.13) to $\chi_l\,\mathrm{e}^{\mathrm{i}\gamma(M-l+1/2)}$ therefore leads to the required result.

The ladder operators do change the normalization, and the usual method of getting the change in the normalization factor does not work because $X_\pm$ are not adjoints. It can, however, be determined by using the matrix elements of sections 4 and 5, and the result is

$$X_\pm \chi_l\,\mathrm{e}^{\mathrm{i}\gamma(M-l+1/2)} = c_\pm \chi_{l\pm1}\,\mathrm{e}^{\mathrm{i}\gamma(M-l+1/2\mp1)}$$

$$c_\pm = -2\left(\frac{(l-\frac{1}{2}\pm\frac{1}{2})(2M-l+\frac{1}{2}\mp\frac{1}{2})(2M-2l+1)[1-\lambda(M-l+\frac{1}{2}\mp\frac{1}{2})^2]}{2M-2l+1\mp2}\right)^{1/2}. \tag{7.15}$$

For $\lambda = 0$ these results agree with those given by Nieto and Simmons (1979).

In the case where $\lambda \neq 0$, the so(3) algebra spanned by L_0, L_+, L_- can be extended to so(4) by considering the additional operators

$$U_0 = -\mathrm{i}\frac{\partial}{\partial x} \qquad U_\pm = \lambda^{-1/2}\,\mathrm{e}^{\pm\mathrm{i}\gamma} g^{-1}\left(f\frac{\partial}{\partial x} \mp \mathrm{i}\frac{\partial}{\partial\gamma}\right). \tag{7.16}$$

One indeed obtains

$$\begin{aligned} &[L_0, U_0] = [L_\pm, U_\pm] = 0 \qquad [L_0, U_\pm] = \pm U_\pm \\ &[L_\pm, U_0] = \mp U_\pm \qquad [L_\pm, U_\mp] = \pm 2U_0 \\ &[U_0, U_\pm] = \mp L_\pm \qquad [U_+, U_-] = -2L_0 . \end{aligned} \tag{7.17}$$

For the two families of Gendenshtein potentials, the operators U_0, $U_\pm$ generalize some operators known for the Rosen–Morse potential (Englefield 1987). When $\lambda = 0$, the operators $U_\pm$ have no counterpart because $f = \pm 1$ leads to $e^{\pm i\gamma} g^{-1}(f\partial/\partial x \mp i\partial/\partial\gamma) \propto K_\pm$. Hence one is only left with the additional operator U_0. One then gets the algebra gl(1)⊕iso(2), where gl(1) is spanned by some linear combination of L_0 and U_0.

8. Conclusion

In this paper we have extended a realization of so(2, 1) to give realizations of either so(2, 2) or so(3, 1) or iso(2, 1). The so(2, 1) can be interpreted as a potential algebra for certain Schrödinger equations, and the larger algebras are then dynamical potential algebras containing operators connecting states of different energy.

The potentials in the Schrödinger equations are the Morse potential given in (6.5), the non-singular Gendenshtein potential in (6.4), and the singular Gendenshtein potential in (6.6). These three cases are naturally distinguished when the potential algebra is extended to a larger algebra. The fact that the same potential algebra is associated with these different problems explains why the three potentials (with the same value of M) support the same energy eigenvalues. We have also written their eigenfunctions in a common form; for example, (5.2)–(5.4) give the normalized eigenfunctions when $\frac{5}{2} < M \leqslant \frac{7}{2}$. The dynamical algebras show that certain operators (see table 4) have matrix elements with a common form which may be obtained using the Wigner–Eckart theorem. Thus the algebraic approach yields unexpected relations between different one-dimensional systems, analogous to supersymmetries. The related problems also include the Rosen–Morse system as a limiting case of the singular Gendenshtein potential.

Our work is actually the first to obtain dynamical algebras associated with the Gendenshtein systems.

Because we can express the operators x and d/dx in terms of the algebra operators (cf table 4), we can obtain (at least in principle) corresponding operators in the three systems. As an example of this (section 7), we have written ladder operators (between energy eigenstates), known for the non-singular Gendenshtein potential, in a form which also applies to the singular case.

Note that the calculation of eigenvalues and matrix elements by algebraic methods presupposes that these quantities exist, and that this assumption cannot usually be verified algebraically. Nevertheless we do find the same eigenvalues and matrix elements exist in the three different cases, provided there are any eigenvalues at all. The singular potential (6.6) supports no eigenvalues if $M > b + \frac{1}{2}$, in contrast to the two non-singular cases. This difference appears by examining the singularities of the functions which are obtained from the algebraic approach. This type of consideration is believed to be important in an algebraic approach to scattering (Biedenharn and Stahlhofen 1987).

Our results for matrix elements are given by (5.12), (5.13) and (5.14), where the operators are given in table 4 in section 6. For the Morse potential (Nieto and Simmons

1979) the operators play a basic role in a definition of coherent states. The matrix elements have therefore been previously calculated using non-algebraic methods, which give a (finite) sum rather than a closed form (Berrondo *et al* 1987).

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